Rate Region of Scheduling a Wireless Network with Discrete Propagation Delays

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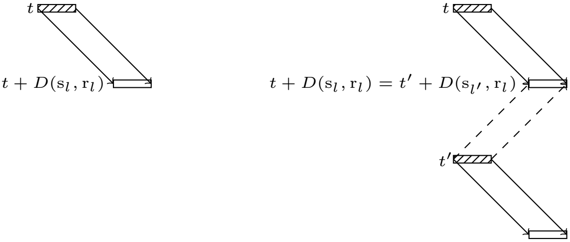
Motivation

- Wireless communication media, e.g., radio, light and sound, all have nonzero signal propagation delays.
- In underwater acoustic communications, the propagation delay can be longer than seconds.
- Previous researches show that taking propagation delay into consideration have signification advantage in throughput and energy consumption.

Network model

- Nodes are indexed by $1, 2, \ldots, N$.
- The signal propagation delay from i to j is $D(i, j) \in \mathbb{Z}^+$.
- A link l is a pair $(\mathrm{s}_l, \mathrm{s}_l)$. Denote $\mathcal{L}$ as a finite set of all the links.
- $\mathcal{I}(l)$ is the collision set of l. When l is active in timeslot t, a *collision occurs* if any $l' \in \mathcal{I}(l)$ is active in the timeslot $t + D(\mathrm{s}_l, \mathrm{r}_l) - D(\mathrm{s}_{l'}, \mathrm{r}_l)$.
- Let $\mathcal{I} = (\mathcal{I}(l), l \in \mathcal{L})$ be the *collision profile* of the network.
- Let $D_{\mathcal{L}}(l, l') = D(\mathrm{s}_l, \mathrm{r}_l) - D(\mathrm{s}_{l'}, \mathrm{r}_l)$ be the link-wise propagation delay



Network model as weighed directed graph

Our network model, denoted by $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D_{\mathcal{L}})$. The network $\mathcal{N}$ can be regarded as a weighted directed graph:

- The set of vertices is specified by $\mathcal{L}$;
- The set of edges is specified by $\mathcal{I}$;
- (l, l') is a directed edge if $l' \in \mathcal{I}(l)$, and has weight $D_{\mathcal{L}}(l, l')$.

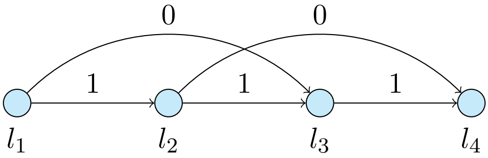
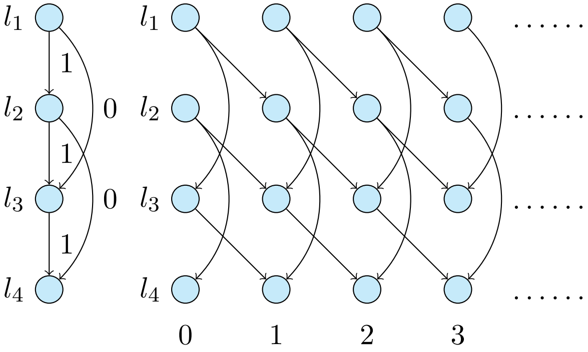


Figure: The graphical representation of $\mathcal{N}^{\text{line}}_{4,1}$.

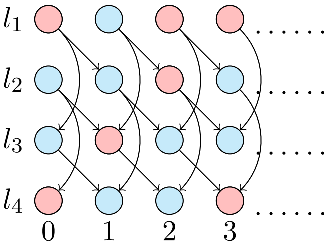
Network model and periodic graph

The *(directed) periodic graph* $\mathcal{N}^\infty$ induced by $\mathcal{N}$.



Periodic graph

A collision free schedule on $\mathcal{N}$ indicates an independent set of the *(directed) periodic graph* $\mathcal{N}^\infty$ induced by $\mathcal{N}$.



$$S = \begin{matrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & \dots \end{bmatrix}.$$

Rate region

- The collection $\mathcal{R}^{\mathcal{N}}$ of all the achievable rate vectors is called the *rate region* of $\mathcal{N}$.
- For a network $\mathcal{N}$, the rate region $\mathcal{R}^{\mathcal{N}}$ can be achieved using collision-free, periodic schedules only.
- Denote D^* as the maximum linkwise propagation delay.

Rate region by subgraphs

Define $\mathcal{N}^T$ as the subgraph of $\mathcal{N}^\infty$ with the vertex set $\mathcal{L} \times \{0, 1, \ldots, T-1\}$.
Define $\mathcal{R}^{\mathcal{N}^T}$ as the convex hull of the rate vectors of all the independent sets of $\mathcal{N}^T$.

Theorem

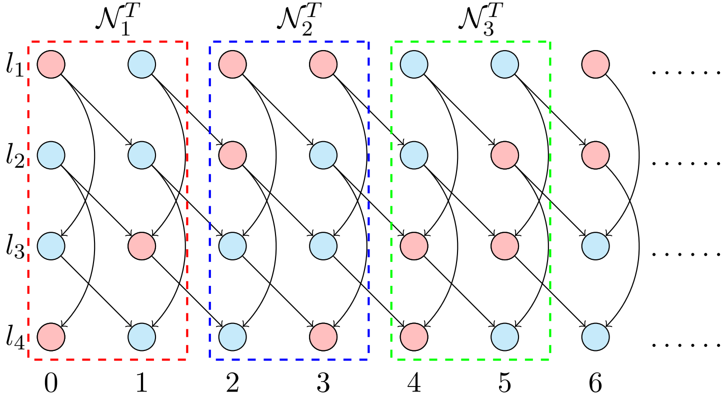
For a network $\mathcal{N}$,

$$\mathcal{R}^{\mathcal{N}} = \mathrm{closure}\left(\cup_{T=1,2,\ldots} \frac{T}{T+D^*} \mathcal{R}^{\mathcal{N}^T} \right),$$

where $\mathrm{closure}(\mathcal{A})$ is the closure of set A.

Conditional independence property

For subgraphs $\mathcal{N}_1^T$, $\mathcal{N}_2^T$ and $\mathcal{N}_3^T$, $\mathcal{N}_3^T$ is conditional independent with $\mathcal{N}_1^T$.
The selection of $\mathcal{N}_{i+1}^T$ is only determined by $\mathcal{N}_i^T$ when $T \geq D^*$.



Scheduling Graphs

A *scheduling graph* is a directed graph denoted by $(\mathcal{M}_T, \mathcal{E}_T)$ defined as follows:

- $\mathcal{M}_T$ is the collection of all independent sets of $\mathcal{N}^T$.
- $\mathcal{E}_T$ is the collection of all independent sets of $\mathcal{N}^{2T}$.

Scheduling Graphs

Example

For $\mathcal{N}_{4,1}^{\text{line}}$, $(\mathcal{M}_1, \mathcal{E}_1)$ has the vertex set $\mathcal{M}_1 = \{v_i, i = 0, 1, \ldots, 8\}$ where

$$v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$v_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_6 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_7 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_8 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

Scheduling Graphs

Example

The adjacency matrix

$$
\begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\end{array}.
$$

Scheduling Graphs and schedules

Theorem

A collision-free schedule S, when $T \geq D^*$ can be represented by a directed path in a schedule $(\mathcal{M}_T, \mathcal{E}_T)$.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_0	1	1	1	1	1	1	1	1	1
v_1	1	1	0	1	1	1	0	0	1
v_2	1	1	1	0	1	1	0	0	1
v_3	1	1	1	1	0	0	1	1	0
v_4	1	1	1	1	1	1	1	1	1
v_5	1	1	0	1	1	1	0	0	1
v_6	1	1	0	0	1	1	0	0	0
v_7	1	1	1	0	0	1	0	0	0
v_8	1	1	1	1	0	1	0	1	0

$$S = \begin{matrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Rate Region by Cycles

- A collision-free schedule S of period forms a closed path in $(\mathcal{M}_T, \mathcal{E}_T)$.
- A closed path can be decomposed into a sequence of (not necessarily distinct) cycles.

Rate Region by Cycles

For a finite directed graph $\mathcal{G}$, we know that $\mathrm{cycle}(\mathcal{G})$ is finite. Define

$$\mathcal{R}^{(\mathcal{M}_T,\mathcal{E}_T)} = \mathrm{conv}(\{R_C : C \in \mathrm{cycle}(\mathcal{M}_T, \mathcal{E}_T)\}),$$

As $(\mathcal{M}_T, \mathcal{E}_T)$ is finite, $\mathrm{cycle}(\mathcal{M}_T, \mathcal{E}_T)$ is finite and hence $\mathcal{R}^{(\mathcal{M}_T,\mathcal{E}_T)}$ is a closed set.

Theorem

For a network $\mathcal{N}$ and any integer $T \geq D^*$, $\mathcal{R}^{\mathcal{N}} = \mathcal{R}^{(\mathcal{M}_T,\mathcal{E}_T)}$.

Dominance properties

- $A \preccurlyeq B$ if all the entries of A are not larger than the corresponding entries of B at the same positions.
- For a set $\mathcal{A}$ with partial order $\succcurlyeq$, we write $\max_{\succcurlyeq} \mathcal{A}$ as the smallest subset $\mathcal{B}$ of $\mathcal{A}$ such that any element of $\mathcal{A}$ is dominated by certain elements of $\mathcal{B}$.

Algorithms for Rate Region: Straightforward approach

- Instead of $\mathcal{E}_T$, we can find $\mathcal{E}^* = \max_{\succcurlyeq} \mathcal{E}_T$, which is the collection of maximal independent sets of $\mathcal{N}^{2T}$.
- Using the Bron–Kerbosch algorithm to enumerate all the maximal independent sets of $\mathcal{N}^{2T}$ to calculate $\mathcal{M}_T$ and $\mathcal{E}^*$, where T can be as small as D^*.
- Using a backtracking algorithm to enumerate all the cycles in $(\mathcal{M}_T, \mathcal{E}_T)$.

An Incremental Approach for Maximal Cycles

- Algorithm 1: Enumerating the maximal paths incrementally.
- Algorithm 2: Finding all cycles dominated by a path.

Algorithm 1: enumerating the maximal paths incrementally.

- $\mathcal{G}_1 = (\mathcal{M}_L^*, \mathcal{U}_0', \mathcal{M}_R^*)$, let $\mathcal{M}_L^*$ (resp. $\mathcal{M}_R^*$) be the collection of B such that $(B, B') \in \mathcal{E}^*$ (resp. $(B', B) \in \mathcal{E}^*$) for certain B'. $\mathcal{E}^* \subset \mathcal{M}_L^* \times \mathcal{M}_R^*$.
- For $k > 1$, we define a directed $(k+1)$-partite graph $\mathcal{G}_k$.

$$\mathcal{G}_k = (\mathcal{M}_L^*, \mathcal{U}_0, \mathcal{V}, \mathcal{U}_1, \mathcal{V}, \ldots, \mathcal{U}_{k-2}, \mathcal{V}, \mathcal{U}_{k-1}', \mathcal{M}_R^*),$$

  where $\mathcal{V} = \{B \wedge B' : B \in \mathcal{M}_R^*, B' \in \mathcal{M}_L^*\}$. $\mathcal{G}_{i+1}$ can be calculated using $\mathcal{G}_i$ and $\mathcal{E}^*$.
- Any k-length maximal path in $(\mathcal{M}_T, \mathcal{E}_T)$ is a path of length k in $\mathcal{G}_k$.

Algorithm 1: example

Example

For the scheduling graph $(\mathcal{M}_1, \mathcal{E}_1)$ of $\mathcal{N}_{4,1}^{\text{line}}$, $\mathcal{G}_1 = (\mathcal{M}_L^*, \mathcal{E}^*, \mathcal{M}_L^*)$ is characterized in Example 2, where $\mathcal{M}_L^* = \mathcal{M}_R^* = \{v_5, v_6, v_7, v_8\}$. For this example, we have $\mathcal{V} = \mathcal{M}_1$, $\mathcal{U}_0$ has an adjacent matrix

$$\begin{array}{c} \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Algorithm 1: example

Example

$\mathcal{U}_1'$ and $\mathcal{U}_2'$ have, respectively, the adjacent matrices

$$\begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccc} v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \text{ and } \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccc} v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Algorithm 1: example

Example

Moreover, we have the adjacent matrix of $\tilde{\mathcal{U}}'$:

$$
\begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array}
\begin{array}{c}
\begin{array}{cccc} v_5 & v_6 & v_7 & v_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\end{array}.
$$

From the adjacent matrices, we see that $\mathcal{U}_1', \mathcal{U}_2' \subset \tilde{\mathcal{U}}'$.

Algorithm 1

Example

Consider the network $\mathcal{N}_{4,1}^{\text{line}}$. For $k = 1, \dots, 4$, we list the number of paths in $\mathcal{G}_k$ and the total number of length-k paths in $(\mathcal{M}_1, \mathcal{E}_1)$ in the following table:

	$k=1$	$k=2$	$k=3$	$k=4$
Number of length-k paths in $\mathcal{G}_k$	6	16	64	180
Number of length-k paths in $(\mathcal{M}_1, \mathcal{E}_1)$	56	363	2357	152633

Algorithm 2: Finding all cycles dominated by a path.

Concluding remarks

- The essential problem: independent sets of a periodic graph.
- Connect periodical independent sets with cycles in scheduling graphs.
- Simplifying the computation costs of enumerating cycles by exploring dominance property.